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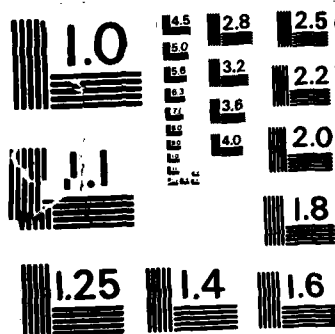
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CENTER FOR BIOLOGICAL INFORMATION PROCESSING  
WHITAKER COLLEGE

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February, 1986

**Perspective Projection Invariants**

**Alessandro Verri and Alan Yuille**

**Abstract:** An important part of stereo vision consists of finding and matching points in two images which correspond to the same physical element in the scene. We show that zeros of curvature of curves are perspective projection invariants and can therefore be used to find corresponding points. They can be used to help solve the registration problem (Longuet-Higgins, 1982) and to obtain the correct depth when a curve enters the forbidden zone (Krol and van de Grind, 1982). They are also relevant to theories for representing image curves. We consider the stability of these zeros of curvature.

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## Perspective Projection Invariants

The goal of stereopsis in computer vision is the recovery of the 3D structure of a scene, given two different images of it. This recovery is preceded by the search for corresponding features of the two images, that is to say features that correspond to the same physical elements in the scene (1,17).

If the two images are taken from two close points of view, they will typically be just slightly different: then standard correlation techniques can identify corresponding elements (2,3,4,5,6,7,8). Once this correspondence has been obtained, triangulation methods allow the 3D reconstruction of the scene. It can be shown that in this case the error made in the reconstruction is large, while if the two images are taken from two very different points of view the error is smaller (9). However when the two images are taken from points of view far apart from each other they will look very different and standard correlation techniques will fail in the search for corresponding elements. This search, therefore, must be guided by new rules, taking into account features having invariant properties under arbitrary perspective projections. Such invariant features will project to corresponding points in the two images.

Epipolar lines in the two images are lines on which corresponding points lie. The projections of a point P in space lie on the plane defined by P and the two camera foci and hence on the two lines defined by the intersection of this plane with the two image planes (see figure 1).

If the epipolar lines are known the matching problem is reduced to a one-dimensional search. Moreover the ordering of edges, or other features, is usually preserved along epipolar lines. Many stereo algorithms make use of the ordering constraint along epipolar lines. This ordering constraint is violated for an object in the "forbidden zone" (14, 15, 16). If a point B lies in the forbidden zone of a point A and the two points are connected by an opaque surface then the two images would see opposite sides of the surface (see figure 2).

To determine the epipolar lines we must find the camera geometry by registering the image. Longuet-Higgins (1982) has described a method for solving the registration problem assuming that three corresponding points in the two images are known. Perspective invariants project to corresponding points in the two images and hence could be used to solve this problem. They could also be used as consistency checks to see if the ordering constraint is violated. When the object enters the forbidden zone the ordering constraint will give false matches. These false matches could be detected if we could find corresponding points.



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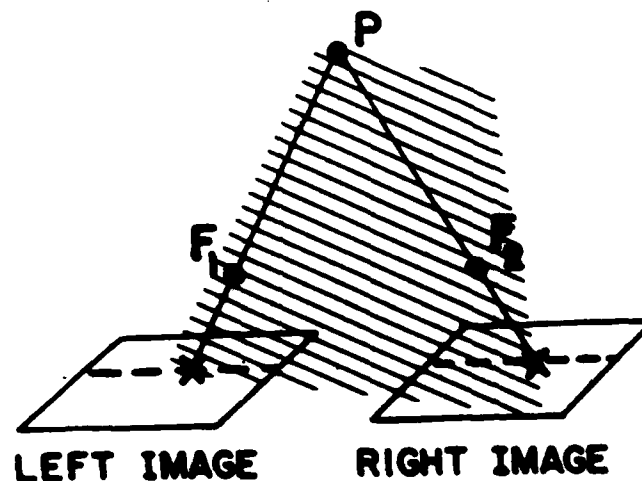


Figure 1. See text.

In this note we prove that points of zero curvature of planar curves are "perspective invariants" and that points of zero curvature of non planar curves are almost always "perspective invariants" with the exception of non generic cases(10). Moreover we briefly analyze the stability in 2D and 3D of such points, considered as zeros of curvature. We show that the zeros of curvature of planar curves are structurally stable for a 2D deformation (one lying on the plane of the curve) but not for a 3D one. Since projections map 3D curves into 2D ones, however, projections of almost planar curves show some stability properties. On the contrary, zeros of curvature of arbitrary 3D curves do not present any simple kind of stability.

Thus zeros of curvature of planar or almost planar curves are features that can be successfully used in solving the "correspondence problem" in stereopsis when the scene is observed from two very different points of view. Since zeros of curvature in two images correspond they can be used to solve the registration problem for stereo (13). The projection invariants of zeros of curvature and the constraint that matches take place between points on corresponding epipolar lines have been used to construct an algorithm able to solve the correspondence problem for arbitrary 2D curves even when the ordering constraint is violated (11).

Recent work on representing image curves involves filtering the curve with gaussians of

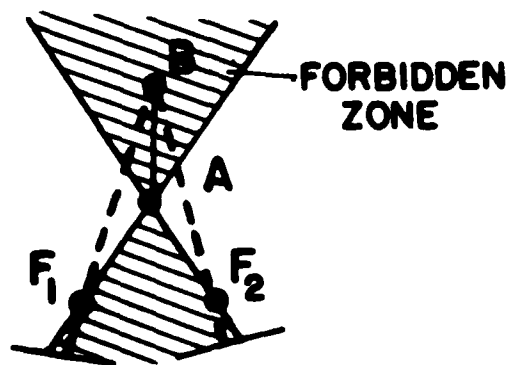


Figure 2. See text.

different sizes to obtain a multiscale description which can be interpreted in terms of a small set of primitives thereby producing a *curvature primal sketch* (18). Marimont (19) considers planar curves projected orthographically onto the viewing plane. He suggests using zeros and extrema of curvature as primitives because of their invariance, and approximate invariance, under change of viewpoint. Zeros of curvature of the curve are moreover preserved when the curve is convolved with a gaussian filter. The results in this paper extend this work by showing that zeros of curvature are invariants even when the curve is non-planar and the projection is perspective.

### Section 1.

Let  $\gamma$  be a curve in the 3D space and  $\gamma_p$  its perspective projection with respect to a focus  $O$  on a plane  $\Phi$ . Let  $f$  be the focal length, the distance between  $O$  and  $\Phi$ .

If we introduce a system of coordinates  $(x, y, z)$  centered on  $O$  then  $\gamma$  will be described by the vector valued function  $\underline{x}(s) = (x(s), y(s), z(s))$  where  $s$ , the arc length of  $\gamma$ , varies in a given interval  $[a, b]$  of  $\mathbb{R}$  and  $\gamma_p$  by  $\underline{x}_p = \underline{x}_p(s)$ . It is easy to see that the following relationship holds between  $\underline{x}_p$  and  $\underline{x}$  (see fig.3)

$$\underline{x}_p(s) = \frac{f}{(x(s) \cdot \underline{n})} \underline{x}(s) \quad (1.1)$$

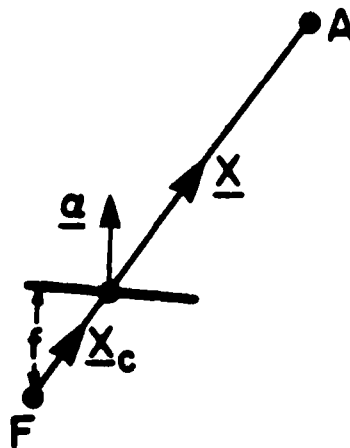


Figure 3. See text.

where  $\underline{a}$  is the unit normal vector to the projection plane  $\Phi$ : notice that  $\underline{x} \cdot \underline{a} \neq 0$  for all  $s$  since we suppose that the perspective projection exists.

From now on  $T(T_p)$  will indicate the unit tangent vector to the curve (projected curve),  $N(N_p)$  the unit normal to the curve (projected curve) and  $B(B_p)$  the unit binormal to the curve (projected curve). Suppose moreover that each component of  $\underline{x}(s)$  is a function twice differentiable with respect to  $s$ .

From elementary differential geometry we have:

$$\kappa(s) = \left\| \frac{d^2 \underline{x}(s)}{ds^2} \right\| \quad (1.2)$$

where  $\kappa(s)$  is the curvature of  $\gamma$  at the point  $\underline{x}(s)$  and:

$$\kappa_p(s_p) = \left\| \frac{d^2 \underline{x}_p(s_p)}{ds_p^2} \right\| \quad (1.3)$$

where  $\kappa_p(s_p)$  is the curvature of  $\gamma_p$  at the point  $\underline{x}_p(s_p)$ , with  $s_p$  being the arc length of  $\gamma_p$ .

Now let us compute  $\kappa_p$ . It is easy to see that:

$$\frac{dx_p}{ds} = \frac{f}{(x \cdot \alpha)^2} J \quad (1.4)$$

where

$$J = \alpha \times (T \times x). \quad (1.5)$$

Since  $s_p$  is the arc length of  $x_p$ ,

$$T_p \cdot T_p = \frac{dx_p}{ds_p} \cdot \frac{dx_p}{ds_p} = 1 = \frac{dx_p}{ds} \cdot \frac{dx_p}{ds} \left( \frac{ds}{ds_p} \right)^2 \quad (1.6)$$

and hence

$$\frac{ds}{ds_p} = \frac{(x \cdot \alpha)^2}{f} \frac{1}{|J|}. \quad (1.7)$$

Notice that (1.7) holds if

$$|J| \neq 0 \quad (1.8a)$$

and that (1.8a) always holds (see (1.3)) unless<sup>1</sup>

$$T \parallel x. \quad (1.8b)$$

Now

$$T_p = \frac{J}{|J|} \quad (1.9)$$

and

$$\kappa_p N_p = \frac{dT_p}{ds_p} = \frac{dT_p}{ds} \frac{ds}{ds_p}. \quad (1.10)$$

Substituting (1.9), (1.7) and (1.3) in (1.10) gives

$$\kappa_p N_p = \frac{d}{ds} \left( \frac{J}{|J|} \right) \frac{(x \cdot \alpha)^2}{f} \frac{1}{|J|} \quad (1.11)$$

but

<sup>1</sup>If this condition is not satisfied then a cusp will typically be generated in  $\gamma_p$ .

$$\frac{d}{ds} \left( \frac{\underline{I}}{|\underline{I}|} \right) = \frac{1}{|\underline{I}|^3} \underline{I} \times \left( \frac{d\underline{I}}{ds} \times \underline{I} \right). \quad (1.12)$$

and so

$$\kappa_p \underline{N}_p = \frac{(\underline{x} \cdot \underline{\omega})^2}{f|\underline{I}|^4} \underline{I} \times \left( \frac{d\underline{I}}{ds} \times \underline{I} \right). \quad (1.13)$$

Notice that

$$\frac{d\underline{I}}{ds} = \kappa \underline{\alpha} \times (\underline{N} \times \underline{x}) = \kappa \underline{L} \quad (1.14)$$

where we set

$$\underline{L} = \underline{\alpha} \times (\underline{N} \times \underline{x}). \quad (1.15)$$

So (1.13) becomes

$$\kappa_p \underline{N}_p = \frac{(\underline{x} \cdot \underline{\omega})^2}{f|\underline{I}|^4} \kappa \underline{M} \quad (1.16)$$

where we set

$$\underline{M} = \underline{I} \times (\underline{L} \times \underline{I}). \quad (1.17)$$

Notice that  $|\underline{M}| = 0$  if and only if  $\underline{I} = \beta \underline{L}$  for some real number  $\beta$ . This can easily be seen from (1.17): looking at (1.5) and (1.14) it is obvious that such a  $\beta$  exists if and only if  $\underline{x}, \underline{N}, \underline{I}$  are coplanar, that is if and only if  $\underline{x} \cdot \underline{N} \times \underline{I} = 0$  (we ignore the possibility that  $\underline{\alpha} \cdot \underline{x} = 0$  since in this case the eye cannot see the point).

Now we can prove the following theorems.

#### Theorem 1.

Let  $\gamma$  be a simple planar curve and  $\gamma_p$  its perspective projection onto a plane  $\Phi$  with respect to a focus  $O$ . Let us use the same notation and hypotheses as above. Then: for every  $s$  for which  $\kappa$  is defined  $\kappa_p$  is also defined. Moreover, whenever there exists an  $s_0$  such that  $\kappa(s_0) = 0$  then  $\kappa_p(s_0) = 0$  and vice versa.

#### Proof

Notice that for a planar curve for which a perspective projection with respect to a focus  $O$  exists,  $\underline{x} \cdot \underline{N} \times \underline{I} \neq 0$  for all  $s$ , since the plane containing the curve does not pass through

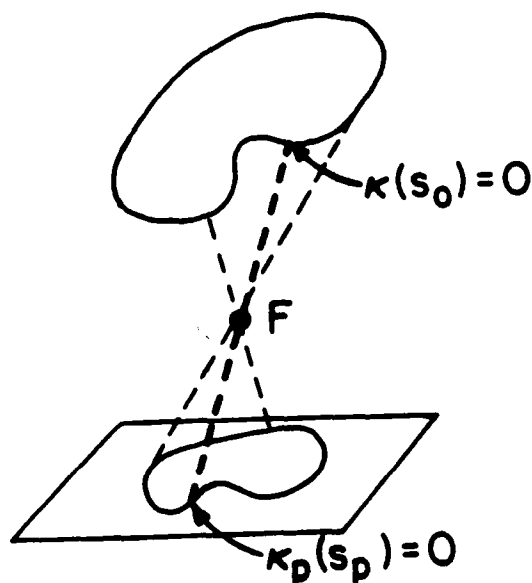


Figure 4. See text.

the point  $O$ . Hence (1.8a) and (1.8b) always hold, thus (1.16) holds as well and the first statement of the theorem is proved.

Moreover from (1.16) it follows immediately that when there exists an  $s_0$  such that  $\kappa(s_0) = 0$  then  $\kappa_p(s_0) = 0$ . The converse is also true since in the right hand side of (1.16)  $(\underline{x} \cdot \underline{\alpha})^2 > 0$  for all  $s$  by hypotheses and  $|\underline{M}| \neq 0$  because as mentioned above  $\underline{x}, \underline{N}, \underline{T}$  cannot be coplanar. So if  $\kappa_p(s_0) = 0$  (1.17) holds if  $\kappa(s_0) = 0$ .

### Theorem 2

Let us suppose that  $\gamma$  is a generic non planar curve and  $\gamma_p$  its perspective projection with respect to a focus  $O$  on a plane  $\Phi$ . Then with the same notation and hypotheses as above: for every  $s$  for which  $\kappa$  is defined and unless  $\underline{T} \parallel \underline{x}$  then  $\kappa_p$  is defined. Moreover whenever there exists  $s_0$  such that  $\kappa(s_0) = 0$  then  $\kappa_p(s_p) = 0$  (see fig.4).

### Proof

Notice that for any curve (1.8a) holds if (1.8b) holds, that is it holds unless  $\underline{T} \parallel \underline{x}$ . Then from the hypotheses of the theorem the proof follows directly from (1.16).

### Theorem 3

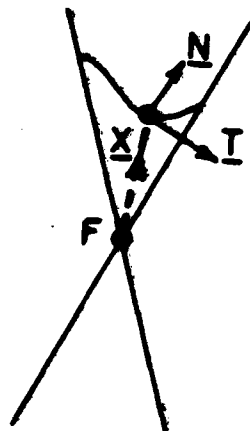


Figure 5. See text.

With the same hypotheses as theorem 2 a necessary and sufficient condition for there to exist an  $s_0$  such  $\kappa_p(s_0) = 0$  while  $\kappa(s_0) \neq 0$  is that  $\underline{x}(s_0) \cdot \underline{N}(s_0) \times \underline{T}(s_0) = 0$ .

#### Proof

The theorem is an direct consequence of the fact that, by the hypotheses of theorem 2, (1.16) holds and that  $|\underline{M}(s_0)| = 0$  if and only if  $\underline{x}(s_0)$ ,  $\underline{N}(s_0)$  and  $\underline{T}(s_0)$  are coplanar.

Theorem 1 says that for any planar curve points of zero curvature are preserved under arbitrary perspective projection: theorem 2 says that for arbitrary curves points of zero curvature are transformed into points of zero curvature of the projected curve. The converse, as shown in theorem 3, is true except for the case in which  $\underline{x}$ ,  $\underline{N}$  and  $\underline{T}$  are coplanar (see fig.5).

. This case, however, is non generic since a small displacement of the focus of projection will destroy the coplanarity.

It is worth noticing that even cusps have the same property of invariance, but as discontinuities of the curvature they are not easily identified in computer implementation (while zeros of curvature are).

#### Section 2.

In this section we discuss the stability of 2D and 3D zeros of curvature. Before starting the discussion let us summarize some elementary results of differential geometry (12).

Let  $\gamma$  be a smooth curve lying on a smooth surface. The component of the curvature vector  $d^2\mathbf{x}/ds^2$  along the direction of the surface normal is called the normal curvature. Therefore if  $\mathbf{n}$  is the unit vector normal to the surface and  $\kappa_n$  the normal curvature, then for every  $s$  we have:

$$\kappa_n = \frac{d^2\mathbf{x}}{ds^2} \cdot \mathbf{n} = \kappa \mathbf{N} \cdot \mathbf{n}. \quad (2.1)$$

Moreover the component of the curvature vector along the unit vector  $\mathbf{u}$ , orthogonal to the surface normal and such that  $\mathbf{u} \cdot (\mathbf{n} \times \mathbf{N}) = 0$  is called the geodesic curvature. It is easy to see that the geodesic curvature at a point is equal to the ordinary curvature times the cosine of the angle  $\theta$  between the osculating plane to the curve (the plane through the point containing the tangent and normal vectors to the curve) and the tangent plane to the surface. Therefore, if  $\kappa_g$  is the geodesic curvature, for every  $s$  we have:

$$\kappa_g = \kappa \cos \theta \quad (2.2)$$

and clearly:

$$\kappa^2 = \kappa_n^2 + \kappa_g^2. \quad (2.3)$$

It is clear that  $\kappa_n$  must have a maximum  $\kappa_1$  and a minimum  $\kappa_2$ , called the first and second principal curvatures respectively, since it is a continuous function on a closed bounded set. Moreover the directions of the two principal curvatures are mutually orthogonal and if  $\phi$  is the angle between the tangent to the curve and the first principal curvature we have:

$$\kappa_n = \kappa_1 \cos^2 \phi + \kappa_2 \sin^2 \phi. \quad (2.4)$$

This relationship will be useful later on, since it suggests examples of surfaces on which no curve can show zeros of  $\kappa_n$  and therefore no zeros of curvature at all.

Consider now the set  $\Gamma$  of curves in the  $x, y$  plane with only one zero of curvature. It can be defined as follows:  $\gamma \in \Gamma$  if its parametric equations  $x = x(s)$ ,  $y = y(s)$  are continuous with derivatives continuous up to every order: moreover

$$x'^2(s) + y'^2(s) = 1 \quad (2.5)$$

for every  $s$  and

$$x''^2 + y''^2 \neq 0 \quad (2.6)$$

for every  $s \neq s_0$ , where  $s_0$  is the point of zero curvature.

At the point  $(x(s_0), y(s_0))$   $\gamma$  has a zero of curvature and in consequence the normal and the binormal to that curve at that point are not well defined. For a planar curve the normal vector to a curve can be defined to be the unit vector lying in the plane perpendicular to the tangent. Since the functions  $x$  and  $y$  have continuous second derivatives the left and right limits of the unit normal as  $s$  goes to  $s_0$  must exist, for when  $s = s_0$ :

$$\underline{N} = \left( \frac{y''(s)}{\sqrt{x''^2(s) + y''^2(s)}}, \frac{-x''(s)}{\sqrt{x''^2(s) + y''^2(s)}} \right). \quad (2.7)$$

Moreover these two limits, not necessarily coincident, allow the existence of the following vector product limits:

$$\begin{aligned} \lim_{s \rightarrow s_0+} T(s) \times N(s) &= B(s_0)_+ \\ \lim_{s \rightarrow s_0-} T(s) \times N(s) &= B(s_0)_- \end{aligned} \quad (2.8)$$

since

$$B(s) = I(s) \times N(s). \quad (2.9)$$

But

$$T(s) \cdot N(s) = 0 \quad (2.10)$$

for every  $s \neq s_0$  and so

$$B(s_0)_+ \cdot B(s_0)_- = \pm 1 \quad (2.10)$$

or equivalently

$$\underline{N}(s_0)_+ \cdot \underline{N}(s_0)_- = \pm 1 \quad (2.11)$$

where  $\underline{N}(s_0)_+$  and  $\underline{N}(s_0)_-$  are respectively the right and left limits of the unit normal for  $s \rightarrow s_0$ .

Expression (2.10) is more useful than (2.11), since for a planar curve the direction of the binormal is constant. So at such points the limit of the binormal can change sign and therefore we can arbitrarily set the value of the sign of the curvature at a given point where  $\kappa \neq 0$  and then detect and define a change of sign whenever the binormal, or rather its limit expression, changes its sign.

We now make use of Morse theory (10). Intuitively if the zero of a function  $f(x)$  is a zero crossing (in other words if the function changes sign at the zero) then a small perturbation of the function will not alter the existence of such a zero, although it may move its position. The zero is then said to be stable. Moreover most zeros of functions are zero crossings. This can be contrasted with the behaviour of a function like  $g(x) = x^4$  which has a zero, but not a zero crossing, at  $x = 0$ . A small perturbation of  $g(x)$ , such as adding an infinitesimal number  $\epsilon$  will destroy this zero. Thus the typical zeros of a typical function will be zero-crossings and stable under perturbations of the function. In Morse theory these intuitions are made precise: typical functions are Morse functions and have transversal zero-crossings which have structurally stable intersections with the zero axis. For our purposes this result means that zeros of curvature are stable if they are zero-crossings.

Note that if we had adopted the definition of curvature as a positive function there would have been a problem of defining transversality of the function itself with respect to zero.

We now consider three dimensional perturbations of non-planar curves. We will consider curves which lie on a cylinder with a fixed cylindrical projection onto the  $x, y$  plane (see fig.6).

Varying the radius of the cylinder gives us a one-parameter family of non-planar curves. Small changes of the radius give rise to small non-planar perturbations of the curve.

Let  $\gamma$  be a planar curve whose parametric equations are  $x = x(t)$ ,  $y = y(t)$ , where  $t$  is the arc length. If we think of every point in the  $x, y$  plane as being the projection of a point lying on a cylinder of radius  $R$  whose principal axis has equation:

$$x = 0, \quad y = y, \quad z = R \quad (2.12)$$

then the equations of the curve  $\gamma_c$  whose cylindrical projection on the  $x, y$  plane gives  $\gamma$  are (see fig.4):

$$x_c = R \sin \frac{x(t)}{R}$$

$$y_c = y \quad (2.13)$$

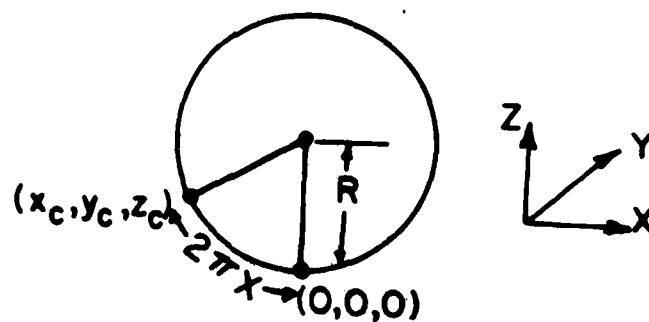


Figure 6. See text.

$$z_c = R - R \cos \frac{x(t)}{R}.$$

Notice that when  $R \mapsto +\infty$  the first order approximation of  $\gamma_c$  gives:

$$x_c = x(t)$$

$$y_c = y(t) \tag{2.14}$$

$$z_c = 0.$$

Therefore when  $R$  is large,  $\gamma_c$  can be seen as a slight 3D perturbation of  $\gamma$ . Let us now compute  $\kappa_c$ , the curvature of  $\gamma_c$ . Differentiating (2.13) gives:

$$x'_c = \cos \frac{x(t)}{R} x'(t)$$

$$y'_c = y'(t) \quad (2.15)$$

$$z'_c = \sin \frac{x(t)}{R} x'(t).$$

Notice that  $t$  is the arc length for  $\gamma_c$  too, since  $x_c'^2 + y_c'^2 + z_c'^2 = 1$ . Differentiating (2.15) we obtain:

$$x_c'' = -\sin \frac{x(t)}{R} \frac{x'^2(t)}{R} + \cos \frac{x(t)}{R} x''(t)$$

$$y_c'' = y''(t) \quad (2.16)$$

$$z_c'' = \cos \frac{x(t)}{R} \frac{x'^2(t)}{R} + \sin \frac{x(t)}{R} x''(t)$$

therefore

$$\kappa_c^2 = \frac{x'^4(t)}{R^2} + x''^2(t) + y''^2(t). \quad (2.17)$$

It is easy to show that (in the obvious notation)

$$\kappa_{c,n} = \frac{x'^2(t)}{R} \quad (2.18)$$

and

$$x''^2(t) + y''^2(t) = \kappa_{c,g}^2. \quad (2.19)$$

Therefore when  $\kappa = 0$ ,  $\kappa_c = 0$  if and only if  $x' = 0$ , i.e. if the tangent to the curve at the zero of curvature is parallel to the cylindric axis. This condition is usually not satisfied, so 2D zeros of curvature disappear for 3D perturbation.

Despite the instability of the 2D zeros of curvature for 3D perturbations, since projection is a mapping from  $R^3$  to  $R^2$ , we can recover in a rough sense some kind of stability for 2D zeros of curvature. In fact every projection of a small 3D perturbation of a planar curve, which had a zero of curvature, shows a zero-crossing of curvature in the neighbourhood of the corresponding projected point. The size of the neighbourhood will depend on the deviation of the surface containing the curve from a plane.

For a generic 3D curve the situation is more complicated since it is not possible to extend the definition of sign of curvature given above. As an example consider a closed smooth curve lying on a cylinder such that there is only one point with a zero of curvature (a curve similar to a distorted sine wave pattern turned in on itself). Explicit computations in this case show that the left and right limits of the binormal at that point have the same direction but opposite sign: however, to define a change of the sign of curvature here leads to a contradictory result since there is a unique such point, by hypothesis, on the curve.

Moreover it is worth noticing that zeros of curvature of 3D curves are extremely uncommon points: in fact looking at the equations it is clear that  $\kappa = 0$  if  $\kappa_g = 0$  and  $\kappa_1/\kappa_2 = -\tan^2 \phi$ . That is if, given the surface containing the curve, the ratio of the two principal curvatures assumes a specific value at a specific point.

### Summary

We have shown that zeros of curvature of all planar curves and of almost all non-planar curves are perspective invariants. The zeros of curvature of planar curves are shown to be structurally stable for deformations of the curve in the plane. This property still holds for small deformations out of the plane but we prove that it vanishes for arbitrary perturbations of the curve. This lack of structural stability puts a limit on the usefulness of these invariants for non-planar curves. Note, however, this is stability for the curve itself: if the curve does have zeros they will be projective invariants.

Zero crossings of the curvature can be used to find corresponding points for stereo matching. Their use is limited since there are, in general, very few of them. They can be best used to help register the image and to detect if an object enters the forbidden zone.

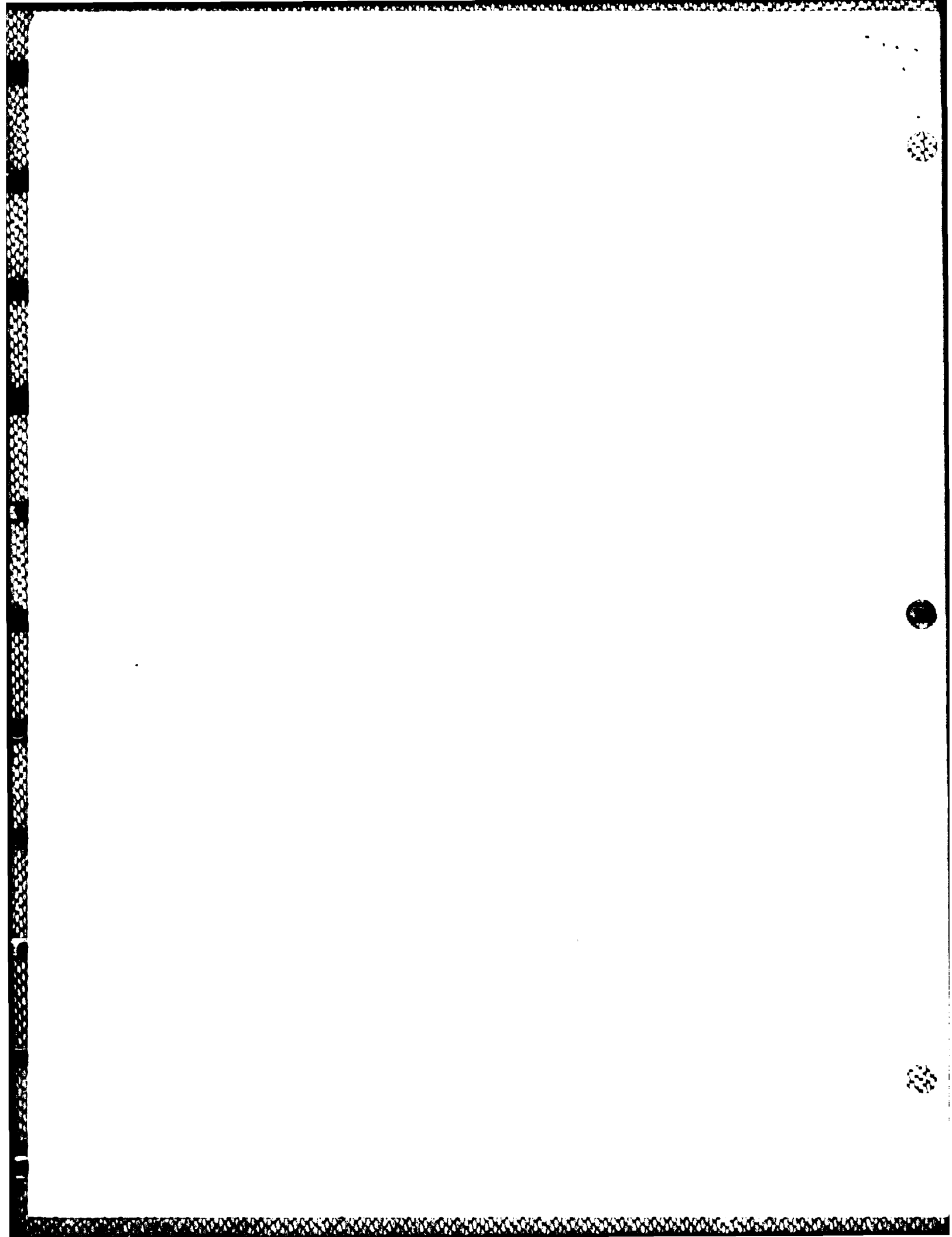
Finally zero crossings are useful for representing image curves because of their invariance under both perspective projection and convolution with gaussian filters.

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